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# On Cubic Interpolatory Splines

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## 1. INTRODUCTION

Schoenberg [3] and de Boor [1] have given an interesting application of splines to histograms by matching the integral means of splines between successive knots with the same means of a given function. Recently Sharma and Tzimbalario [4] have studied quadratic splines with similar matching conditions of integral means. Our object is to study the corresponding problem for cubic splines. Elsewhere Subotin [5] has also considered integral means of cardinal splines in connection with a different problem. An interesting approach to the study of splines which satisfy conditions involving functionals is due to Varga [6].

## 2. EXISTENCE AND UNIQUENESS (Equidistant Knots)

Let $m$, $n$ be positive integers and let $\pi_m$ denote the class of all polynomials of degree $\leqslant m$ and $\Delta = \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}$ be a partition of $[0, 1]$ such that $x_{i+1} - x_i = h$, $i = 0, 1,..., n - 1$. Let $S_p{}^0(m, \Delta) = \{s(x)$: $s(x) \in C^{m-1}[0, 1]$, $s(x) \in \pi_m$, $x \in [x_i, x_{i+1}]$, $i = 0, 1,..., n - 1\}$. Let $f(x)$ be a 1-periodic locally integrable function with respect to a nonnegative measure $d\mu(x)$ where

$$\mu(x + h) - \mu(x) = K, \tag{2.1}$$

for relevant values of $x$. We shall prove the following.

THEOREM 1. *Let $f \in C^2[0, 1]$ be a 1-periodic locally integrable function with respect to a nonnegative measure $d\mu$ satisfying* (2.1). *Suppose further that either*

$$\int_0^h \alpha(x)\, d\mu > 0 \qquad or \qquad \int_0^h \alpha(h - x)\, d\mu > 0, \tag{2.2}$$



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*where* $\alpha(x) = 3x^3 - 6hx^2 + h^3$. *Then there exists a unique* $s(x) \in S_p{}^0(3, \Delta)$, *satisfying the following conditions*:

$$\int_{x_{i-1}}^{x_i} \{f(x) - s(x)\}\, d\mu = 0, \qquad i = 1, 2,..., n, \tag{2.3}$$

$$s^{(r)}(0) = s^{(r)}(1), \qquad r = 0, 1, 2. \tag{2.4}$$

It is interesting to observe that condition (2.3) reduces to different interpolating conditions by suitable choices of $\mu(x)$. Thus, if $\mu$ has a jump of 1 at $x_{i-1} + \lambda h$, $0 < \lambda < 1$, $i = 1,..., n$, then in view of (2.1), condition (2.3) becomes the interpolating condition for that point. This leads to the case considered in [2]. When $\mu(x)$ satisfies (2.1) and has a single jump at one of the endpoints of each mesh, then (2.3) reduces to interpolating condition at the knots.

*Proof of Theorem* 1. For $i = 1, 2,..., n$, set

$$\int_0^h (h - x)^r\, d\mu = h^r A(r); \qquad \int_0^h x^r\, d\mu = h^r B(r); \quad r = 1, 2, 3;$$

$$F_i = \int_{x_{i-1}}^{x_i} f\, d\mu; \qquad H = \int_0^h d\mu.$$

We first observe that as a consequence of (2.1), we have for all $i$ and $r$

$$\int_{x_{i-1}}^{x_i} (x_i - x)^r\, d\mu = h^r A(r);$$
$$\int_{x_{i-1}}^{x_i} (x - x_{i-1})^r\, d\mu = h^r B(r); \qquad \int_{x_{i-1}}^{x_i} d\mu = H. \tag{2.5}$$

Writing $s''(x_i) = M_i$, $i = 0, 1,..., n$, we now proceed to obtain a representation of $s(x)$ in terms of $M_i$'s. Since $s''(x)$ is linear in $x_{i-1} \leqslant x \leqslant x_i$, we have for $x \in [x_{i-1}, x_i]$

$$s'(x) = -(1/2h)\, M_{i-1}(x_i - x)^2 + (1/2h)\, M_i(x - x_{i-1})^2 + c_i \tag{2.6}$$

for $i = 1,..., n$. Here $c_i$'s are the constants to be determined by continuity requirement of $s'(x)$. Thus,

$$M_i h = c_{i+1} - c_i\,. \tag{2.7}$$

Integrating (2.6) and using the continuity requirement for $s(x)$, we have

$$\begin{aligned} s(x) = {} & (1/6h)[M_{i-1}(x_i - x)^3 + M_i(x - x_{i-1})^3] \\ & - \tfrac{1}{2} c_i[(x_i - x) - (x - x_{i-1})] + d_i \end{aligned} \tag{2.8}$$

and

$$h(c_i + c_{i+1}) = 2(d_{i+1} - d_i). \tag{2.9}$$

We now use hypothesis (2.3) and consequences (2.5) to get

$$6F_i = h^2[A(3)\, M_{i-1} + B(3)\, M_i] - 3hc_i[A(1) - B(1)] + 6d_iH. \quad (2.10)$$

Thus, we have from (2.9) and (2.10),

$$\begin{aligned} &h^2B(3)\, M_{i+1} + h^2[A(3) - B(3)]\, M_i - h^2A(3)\, M_{i-1} \\ &\quad - 3h[A(1) - B(1)](c_{i+1} - c_i) + 3hH(c_i + c_{i+1}) = 6(F_{i+1} - F_i). \end{aligned}$$

Using (2.7), we get

$$\begin{aligned} &h^2B(3)\, M_{i+1} + h^2[A(3) - B(3) - 3A(1) + 3B(1) - 3H]\, M_i \\ &\quad - h^2A(3)\, M_{i-1} + 6hHc_{i+1} = 6(F_{i+1} - F_i). \end{aligned}$$

Thus, observing that $A(1) + B(1) = H$, we finally have

$$\begin{aligned} &B(3)\, M_{i+1} + [A(3) - 2B(3) + 6B(1)]\, M_i \\ &\qquad + [-2A(3) + B(3) + 6A(1)]\, M_{i-1} + A(3)\, M_{i-2} \\ &\quad = 6(F_{i+1} - 2F_i + F_{i-1})/h^2. \end{aligned} \quad (2.11)$$

We now prove that Eq. (2.11) has a unique system of solutions under the assumptions of our theorem. The coefficients of $M_{i+1}$ and $M_{i-2}$ in (2.11) are clearly nonnegative. Since $B(1) \geqslant B(3)$ and $A(1) \geqslant A(3)$ hence the coefficients of $M_i$ and $M_{i-1}$ are also nonnegative. Now the difference of the coefficient of $M_{i-1}$ over the sum of the coefficients of $M_{i+1}$, $M_i$, $M_{i-2}$ is

$$\begin{aligned} &2[-2A(3) + B(3) + 3(A(1) - B(1))] \\ &\quad = 2h^{-3} \int_0^h (3x^3 - 6hx^2 + h^3)\, d\mu. \end{aligned}$$

Thus, if the first part of (2.2) holds, the matrix of the coefficients of $M_i$'s in (2.11) becomes diagonally dominant and unique $M_i$'s are determined. Similarly, it may be shown that if the second part of (2.2) holds, then the coefficient of $M_i$ dominates the sum of the coefficients of $M_{i-1}$, $M_{i-2}$, $M_{i+1}$, and again (2.11) has unique solutions. This proves our theorem.

*Remark* 1. Let us now consider $\mu$ to be a function satisfying (2.1) and having a jump at the points $x_i + \lambda h$ where $0 < \lambda < 1$. In this case one of the conditions in (2.2) reduces to

$$\alpha(\lambda) = 3\lambda^3 - 6\lambda^2 + 1 > 0.$$

Since $\alpha(\frac{1}{3}) > 0$, $\alpha(\frac{1}{2}) < 0$, we arrive in particular at the result of Meir and Sharma [2].

The diagonal dominant property which is ensured by (2.2) has been used to prove Theorem 1. The following theorem which we shall now prove takes care of some other cases which are not covered by Theorem 1.

THEOREM 2. *For every odd $n$, Theorem 1 continues to hold if we assume the conditions*

$$\int_0^h (6hx^2 - 4x^3 - h^3)\, d\mu = 0 \qquad and \qquad H > 0 \tag{2.12}$$

*in place of* (2.2).

*Remark* 2. When $\mu$ has a jump at $x_i + \frac{1}{2}h$, $i = 0, 1, \ldots, n-1$, then clearly (2.12) is satisfied whereas (2.2) is not valid. Thus, Theorem 2 is applicable in this case and it corresponds to interpolation at the midpoints for odd number of knots.

*Proof of Theorem* 2. We rewrite Eq. (2.11) determining $M_i$'s as follows

$$\begin{aligned} B(3)\, M_{i+1} &+ [A(3) - 3B(3) + 6B(1)]\, M_i + A(3)\, M_{i-1} \\ &+ B(3)\, M_i + [B(3) - 3A(3) + 6A(1)]\, M_{i-1} + A(3)\, M_{i-2} \\ &= 6(F_{i+1} - 2F_i + F_{i-1})/h^2. \end{aligned} \tag{2.13}$$

Now, we observe that under condition (2.12) the coefficient of $M_i$ in the first line of (2.13) is the same as the coefficient of $M_{i-1}$ in the second line of (2.13). Thus, using the periodicity and summing up Eq. (2.13) after multiplying with $(-1)^{i-1}$, we have

$$\begin{aligned} B(3)\, M_{n+1} &+ [A(3) - 3B(3) + 6B(1)]\, M_n + A(3)\, M_{n-1} \\ &= 3h^{-2} \sum_{i=1}^{n} (-1)^{i-1}(F_{i+1} - 2F_i + F_{i-1}), \end{aligned}$$

since $n$ is odd. Finally, therefore, we have

$$\begin{aligned} B(3)\, M_{i+1} &+ [A(3) - 3B(3) + 6B(1)]\, M_i + A(3)\, M_{i-1} \\ &= 3h^{-2} \left[ \sum_{j=1}^{i} - \sum_{j=i+1}^{n} \right] (-1)^{i+j}(F_{j+1} - 2F_j + F_{j-1}). \end{aligned} \tag{2.14}$$

Now it is clear that the coefficients of $M_{i+1}$, $M_i$, and $M_{i-1}$ in (2.14) are nonnegative and the difference of the coefficient of $M_i$ over the sum of coefficients of $M_{i+1}$ and $M_{i-1}$ is

$$6B(1) - 4B(3),$$

which is positive by virtue of condition (2.12) and the fact that $B(1) \geqslant B(2)$. Thus the system of equations (2.14) has a unique solution.

## 3. Error Estimates

In what follows, $w(f; h)$ will denote the modulus of continuity of $f$. Let $f(x) \in C^2[0, 1]$ and let $s(x)$ be the cubic spline of Theorem 1 with $e(x) = s(x) - f(x)$, $e_i'' = e''(x_i)$.

We shall prove:

THEOREM 3. *The following error-bound holds*:

$$\| e'' \| \leqslant [1 + \{((9H + 12B(1) + 4B(3))/\delta\}]\, w(f; h),$$

*where* $\delta = \int_0^h \alpha(x)\, d\mu$ *or* $\int_0^h \alpha(h - x)\, d\mu$ *according as*

$$\int_0^h \alpha(x)\, d\mu > 0 \qquad \text{or} \qquad \int_0^h \alpha(h - x)\, d\mu > 0.$$

*Proof of Theorem* 3. From (2.11), we obtain the system of equations for $e_i''$ as follows

$$\begin{aligned} B(3)\, e_{i+1}'' &+ [A(3) - 2B(3) + 6B(1)]\, e_i'' \\ &+ [-2A(3) + B(3) + 6A(1)]\, e_{i-1}'' + A(3)\, e_{i-2}'' \\ &= 6h^{-2}(F_{i+1} - 2F_i + F_{i-1}) - B(3)\, f_{i+1}'' - A(3)\, f_{i-2}'' \\ &\quad - [A(3) - 2B(3) + 6B(1)]\, f_i'' - [-2A(3) + B(3) + 6A(1)]\, f_{i-1}''\,. \end{aligned}$$

Now using the results that $f(x) = f_j + (x - x_j) f_j' + \frac{1}{2}(x - x_j)^2 f''(\xi_j)$ and $f'(x) = f_j' + (x - x_j) f''(\eta_j)$, where $\xi_j$ and $\eta_j$ lie in appropriate intervals, we obtain the right-hand side of (3.1) as

$$\begin{aligned} R = &-B(3)\, f_{i+1}'' - [A(3) - 2B(3) + 6B(1)]\, f_i'' \\ &- [-2A(3) + B(3) + 6A(1)]\, f_{i-1}'' - A(3)\, f_{i-2}'' \\ &+ 6Hf''(\theta_{i-1}) + 3[B(2)\, f''(\xi_i) - A(2)\, f''(\xi_{i-1})] \\ &- 3[B(2)\, f''(\eta_{i-1}) - A(2)\, f''(\eta_{i-2})], \end{aligned}$$

where $y_i \in [x_i, x_{i+1}]$ for $y = \xi, \eta, \theta$. Rearranging the terms suitably we get

$$| R | \leqslant [9H + 12B(1) + 4B(3)]\, w(f; h).$$

Now, following the standard argument based on the diagonal dominance property, we get

$$\max_j | e_j | \leqslant [9H + 12B(1) + 4B(3)]\, w(f; h)/\delta,$$

where $\delta$ is as defined in the statement of Theorem 3. Since $e''(x)$ is linear in each interval $[x_{i-1}, x_i]$, Theorem 3 follows by the standard reasoning used elsewhere.

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